Basic Schubert Calculus (Part 2)

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Outline

Chow Cohomology Ring of $\mathcal{F}_n$

Monk’s Formula

Schubert Polynomials via Transition Equations

The Complete Solution to Hilbert’s 15 Problem
Schubert Problems

Fix $d$ permutations and $d$ reference flags. The intersection of the corresponding Schubert varieties is a variety

$$Y = X_{w_1}(R_1^1) \cap X_{w_2}(R_2^2) \cap \cdots \cap X_{w_d}(R_d^d).$$

Each of the irreducible components of $Y$ will be rationally equivalent to translates of Schubert varieties.

**Generalized Schubert Problem.** How many Schubert varieties of each type appear as irreducible components of $Y$?
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**Modern Solution.** Use the theory of Chow Rings and Schubert polynomials to count the number of each type.
Historical Perspective

**Thm.** (Ehresmann, 1934) The complete flag manifold $\mathcal{F}_n \approx GL_n/B$ has a partition into Schubert cells $C_w$ for $w \in S_n$ such that $X_w \subset C_w = \bigcup_{v \leq w} C_v$ where $\leq$ is "Bruhat order". Also, the Poincaré polynomial of $H^*(GL_n/B, \mathbb{Z})$ is given by mapping $q \to q^2$ in $[n]_q! = \prod_{k=1}^n (1 + q + q^2 + \cdots + q^{k-1})$.

**Thm.** (Borel, 1953) $H^*(GL_n/B, \mathbb{Z})$ is isomorphic as a ring to the coinvariant algebra $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle$ where $e_k$ is the $k^{th}$ elementary symmetric function.

**Thm.** (Chow, 1956) $H^*(GL_n/B, \mathbb{Z})$ has a basis given by classes related to Schubert varieties $\{[X_w] \mid w \in S_n\}$ and multiplication has an interpretation in terms of intersection of Schubert varieties.
“Intersection Theory” (Fulton, 1998) completed the details on the isomorphism between the Chow ring and the cohomology ring.

**Chow Ring of $\mathcal{F}_n$.**

- Each variety $X \subset \mathcal{F}_n$ determines an equivalence class $[X]$, where $[X] = [Y]$ if $X$ and $Y$ are rationally equivalent varieties, e.g. $[X_w(B\bullet)] = [X_w(R\bullet)] = [X_w]$.

- Addition of classes is just formal addition.

- If $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_d \subset \mathcal{F}_n$ is a variety with $d$ irreducible components then $[Z] = [Z_1] + [Z_2] + \cdots + [Z_d]$. 
Chow Ring of $\mathcal{F}_n$.

- The classes $\{[X_w] : w \in S_n\}$ form a linear basis for a vector space over $\mathbb{Q}$ graded by codimension. (Chow's thm)

- Multiplication rule: For $R_\bullet$ and $B_\bullet$ in general position

$$[X_u] \cdot [X_v] = [X_u(R_\bullet) \cap X_v(B_\bullet)] = \sum c_{uv}^w [X_w].$$

The structure constant $c_{uv}^w$ is the number of irreducible components of $X_u(R_\bullet) \cap X_v(B_\bullet)$ rationally equivalent to $X_w(G_\bullet)$. 
Intersection Theory (Secret Sauce!)

Let $w_0 = n(n - 1) \ldots 321 \in S_n$.

- By intersecting sets:
  $$[X_w] \cdot [X_{w_0w}] = [X_w(E) \cap X_{w_0w}(F)] = \{w\} = [X_{id}]$$

- The Schubert variety $X_{id}$ is a single point in $\mathcal{F}_n$.

- Perfect pairing:
  $$[X_u(E)] \cdot [X_v(F)] \cdot [X_{w_0w}(G)] = c_{uv}^w [X_{id}]$$

Structure constants: $c_{uv}^w = \#X_u(E) \cap X_v(F) \cap X_{w_0w}(G) \in \mathbb{Z}_{\geq 0}$. Assuming all flags $E$, $F$, $G$ are in sufficiently general position.
Summary. Chow ring structure constants are the answers to 0-dimensional Schubert problems, and they can be computed via the ring homomorphism with $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle$.

Goal. Map each class $[X_w]$ to a polynomial $\mathbb{Z}[x_1, \ldots, x_n]$ of degree $\text{codim}(X_w) = \binom{n}{2} - \text{inv}(w)$.

Notation. $[X_w] \mapsto \mathcal{G}_{w_0 w}(x_1, \ldots, x_n)$ (Schubert polynomial)
Monk’s Perspective (1959)

Monk’s Introduction says:

*The present work was suggested by a comparison of the results of Ehresmann and Borel. Its purpose is to study the geometrical properties of the flag manifold, mainly by the methods of classical algebraic geometry, and to exhibit the relation between the different forms of the basis obtained by these authors.*

**Thm.** (Monk, 1959) For $v \in S_n$ and $s_r = (r \leftrightarrow r + 1)$

$$G_v \cdot G_{s_r} = \sum_{i \leq r < j : \text{inv}(vt_{ij}) = \text{inv}(v) + 1} G_{vt_{ij}}.$$

All Schubert classes are determined by this formula along with the “natural choices”: $G_{s_r} = -(x_1 + x_2 + \ldots + x_r)$, but signs and multiplicities occur.
Monk’s Formula

**Thm.** (Monk, 1959) For \( v \in S_n \) and \( s_r = (r \leftrightarrow r + 1) \)

\[
\mathcal{G}_v \cdot \mathcal{G}_{s_r} = \sum_{i \leq r < j: \text{inv}(vt_{ij}) = \text{inv}(v)+1} \mathcal{G}_{vt_{ij}}.
\]

**Proof Sketch.** \( \mathcal{G}_{s_r} \) corresponds with the codimension 1 Schubert variety indexed by \( w_0s_r = n \ldots r(r + 1) \ldots 321 \). Intersections with \( X_{w_0s_r} \) can be computed by analyzing flags concretely.

Note, intersections with \( X_{w_0} \) are trivial since \( X_{w_0} = F_n \implies \mathcal{G}_{i_d} = 1 \).
A Recurrence Relation for Schubert Polynomials

**Thm.** (Monk, 1959) For $v \in S_n$ and $s_r = (r \leftrightarrow r + 1)$

$$S_v \cdot S_{s_r} = \sum_{i \leq r < j : \text{inv}(vt_{ij}) = \text{inv}(v) + 1} S_{vt_{ij}}.$$

**Alternative Choice.** $S_{s_r} = x_1 + x_2 + \ldots + x_r$ leads to a recurrence for Schubert polynomials with all nonnegative coefficients!

**Transition Equation.** Set $S_{id} = 1$. For all $w \neq id$, let $(r < s)$ be the lex largest pair such that $w(r) > w(s)$. Then

$$S_w = x_r S_{w_{tr_s}} + \sum S_{w'}$$

where the sum is over all $w'$ such that $\text{inv}(w) = \text{inv}(w')$ and $w' = wt_{rs} t_{ir}$ with $0 < i < r$. Call this set $T(w)$. 
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**Example.** If $w = 7325614$, then $r = 5$, $s = 7$

$$S_w = x_5 S_{7325416} + S_{7425316} + S_{7345216}$$

So, $T(w) = \{7425316, 7345216\}$. 

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**Classical Transition Equation**
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**Example.** If $w = 7325614$, then $r = 5$, $s = 7$

$$S_w = x_5 S_{7325416} + S_{7425316} + S_{7345216}$$

... 

$$= x_1^6 x_2^2 x_3^2 x_4 x_5^2 + x_1^6 x_2^2 x_3^2 x_4 x_5^2 + x_1^6 x_2^3 x_3 x_4 x_5^2$$

$$x_1^6 x_2^2 x_3^2 x_4 x_5 + x_1^6 x_2^3 x_3 x_4 x_5 + x_1^6 x_2^3 x_3^2 x_4 x_5.$$ 

**Observations.** The reverse lex largest term $x_1^6 x_2^2 x_3 x_4^2 x_5^2$ corresponds with the exponent vector $(6, 2, 1, 2, 2, 0, 0)$. 

$$C_{7325614}(R_\bullet) = \left\{ \begin{pmatrix} * & * & * & * & * & * & * & 1 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 1 & 0 & 0 \\ * & 0 & 0 & * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} : * \in \mathbb{C} \right\}$$
Schubert Polynomial Basis

**Def.** The *code of the permutation* $w \in S_n$ is

\[
\text{code}(w) = (c_1, c_2, \ldots, c_n)
\]

where $c_i$ is the number of inversions ($i < j$) such that $w_i > w_j$ with $i$ fixed.

The code of a permutation gives a bijection from $S_n$ to *sub-staircase vectors* in

\[
\{0, \ldots, n-1\} \times \{0, \ldots, n-2\} \times \cdots \{0\}.
\]

**Observation.** Up to trailing fixed points, every monomial in

\[
\mathbb{Z}[x_1, x_2, \ldots]
\]

corresponds with a unique Schubert polynomial in $S_\infty$ with that leading term. Therefore, Schubert polynomials are a basis for all polynomials in $\mathbb{Z}[x_1, x_2, \ldots]$.
**Schubert Solutions.** Given permutations $u, v \in S_\infty$, find the expansion

$$\mathcal{G}_u \mathcal{G}_v = \sum c_{w_0u,w_0v}^w \mathcal{G}_w$$

Therefore, the structure constants $c_{uv}^w$ can be found by linear algebra and the Transition Equation!

**Next Frontier.** What is the best way to find the $c_{uv}^w$?
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**Warning.** I never say “It is an open problem to find a combinatorial interpretation for the $c_{uv}^W$’s”. They already count something!

Plus, Izzet Coskun’s claimed his Mondrian tableaux fit that description. Can his algorithm be made more clear?
Pipe Dreams

**History.** Schubert polynomials were originally defined by Lascoux-Schützenberger early 1980’s. Via work of Billey-Jockusch-Stanley, Fomin-Stanley, Fomin-Kirillov, Billey-Bergeron in the early 1990’s we know the following equivalent definition.

**Thm.** For \( w \in S_n \), \( \mathfrak{S}_w(x_1, x_2, \ldots, x_n) = \sum_{D \in RP(w)} x^D \) where \( RP(w) \) are the *reduced pipe dreams* for \( w \), aka *rc-graphs*.

**Example.** A reduced pipe dream \( D \) for \( w = 314652 \) where
\[
x^D = x_1^3x_2x_3x_5.
\]

![Pipe Dream Diagram](imageURL)
**Pipe Dream Transition Equation**

**Bijective Proof.** (Billey-Holroyd-Young, 2019) Based on David Little’s bumping algorithm, the bounded bumping algorithm applied to the \((r, s)\) crossing in a reduced pipe dream for \(w\) to bijectively prove

\[
\mathcal{G}_w = x_r \mathcal{G}_{w_{tr}} + \sum_{w' \in T(w)} \mathcal{G}_{w'}.
\]

Equivalently, we have a bijection

\[
RP(w) \longrightarrow RP(w_{tr}) \cup \bigcup_{w' \in T(w)} RP(w').
\]

Note: This algorithm is not the same as the one used in Billey-Bergeron (1993) to bijectively prove Monk’s formula!
Hilbert’s 15th Problem

Mathematical Problems by Professor David Hilbert.
Lecture delivered at the ICM, 1900. (Bull.AMS)

15. Rigorous Foundation of Schubert’s Enumerative Calculus.

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert† especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

Although the algebra of to-day guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.
Conclusion

Many Thanks!